Simple non-perturbative solution for MHD viscous flow due to a shrinking sheet

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1. Introduction

The boundary layer problem due to a stretching sheet has relevance to extrusion problems and has received considerable interest. However, works on the flow problems due to a shrinking sheet are scarce. Wang [1] was the first to study the unsteady viscous flow induced by a shrinking film. The proof of the existence and (non)uniqueness, the exact solutions, both numerical and in closed form, are given by Miklavcic and Wang [2] for the steady viscous hydrodynamic flow due to a shrinking sheet for a specific value of the suction parameter. Miklavcic and Wang [2] concluded that the solution for shrinking sheets may not be unique at certain suction rates for both two-dimensional and axisymmetric flows. Sajid and Hayat [3] studied the magnetohydrodynamic (MHD) viscous flow due to shrinking sheet for the cases of two-dimensional and axisymmetric shrinking. In [4], Sajid et al. studied the MHD rotating flow of a viscous fluid over a shrinking sheet. They showed that for the shrinking surface the stable and convergent solutions are possible only for MHD flows. Very recently, Wang [5] investigated the stagnation flow towards a shrinking sheet and found for the first time that non-alignment of the stagnation flow and the shrinking of the sheet destroys the symmetry and complicates the flow field.

In [3,4], the explicit analytic solutions were derived by the homotopy analysis method (HAM) which was first developed by Liao [6] for general nonlinear problems. The early applications of HAM to some classical nonlinear fluid dynamics prob-
lems like the Blasius’ viscous flows were presented in [7–10]. Recently, HAM has also been adopted by Ayub et al. [11,12] for
the viscoelastic and third-grade fluid flow problems. The applications of HAM to Oldroyd and micropolar fluid flow problems
were presented in [13–15]. Hayat and Javed [16] solved the boundary layer flow problem in a porous medium by HAM.

Unlike HAM, the Adomian decomposition method (ADM) [17] when applied to boundary layer problems is of the type semi-
analytic-numeric method. The much simpler analytic ADM has been successfully applied to a wide class of linear and nonlinear
differential equations [18–21]. Applications of ADM to boundary-layer equations are given in [22] for the classical Blasius’ equa-
tions [23] for the laminar boundary layer equation of Marangoni convection in In–Ga–Sb system and [24] for the boundary
layer equation of viscous flow due to a moving sheet. A class of laminar boundary layer equations was also investigated using the
ADM in [25,26]. Recently, Awang Kechil and Hashim [27] were the first to extend the applicability of the ADM to an unsteady
boundary layer problem over an impulsively stretching sheet. The first successful application of the ADM to a system of coupled 2-by-2
nonlinear ordinary differential equations of free-convective boundary layer was presented by Awang Kechil and Hashim [28].
The applicability of ADM has recently been extended by Awang Kechil and Hashim [29] to study the boundary layer flow over a
nonlinearly stretching sheet with chemical reaction and magnetic field governed by a system of 4-by-4 ordinary differential
equations. An approximate analytical solution for MHD stagnation-point flow in porous media was presented in [30].

The aim of this paper is to present a simple recursive algorithm based on the ADM which produces the series solution of the
MHD viscous flow due to shrinking sheet. The difficulty of the condition at infinity is overcome by the use of Padé approximants
[31]. The velocity profiles given by the ADM are shown to be in good agreement with the HAM solutions given in [3].

2. Mathematical model

The governing equations for MHD viscous flow are derived from the three-dimensional Navier–Stokes equations which
consist of a continuity equation which reflects a viscous incompressible flow and three momentum equations written in
terms of the components u, v and w. For MHD flow, the additional terms of the magnetic field, B0, are included as the last
term of the first two momentum equations (2), (3) as given in [3].

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, 
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\sigma B_0^2}{\rho} u, 
\]

\[
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\sigma B_0^2}{\rho} v, 
\]

\[
\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), 
\]

subject to

\[
u = -a(x - 1)y, \quad w = -W \text{ at } y = 0, \]

\[
u \to 0 \text{ as } y \to \infty, 
\]

where \(\nu = \mu/\rho\) is the kinematic viscosity, \(\mu\) is the dynamic viscosity and \(\sigma\) is the electrical conductivity, \(a > 0\) is the shrinking
constant and \(W\) is the suction velocity. The cases \(m = 1\) and \(m = 2\) correspond to shrinking sheets in \(x\)- and \(y\)-directions,
respectively. By suitable similarity transformations, Sajid and Hayat [3] transformed the above equations to the following
nonlinear boundary value problem:

\[
\frac{f'''}{M^2 f'} - \frac{f''}{M^2} + mff' = 0, 
\]

subject to

\[
\frac{f'}{M^2} = s, \quad f' = -1 \text{ at } \eta = 0, \]

\[
\frac{f'}{M^2} \to 0 \text{ as } \eta \to \infty, 
\]

where \(s = W/m \sqrt{\mu v}\) and \(M^2 = \sigma B_0^2/\rho a\).

3. ADM solution and discussion

We shall in this section demonstrate the simple application of the ADM [17] to obtain an approximate analytical solution
of (7)–(9). First, we write (7) in the operator form,

\[
L f = \frac{f'''}{M^2 f'} - \frac{f''}{M^2} + mff', 
\]

where \(L = d^3/d\eta^3\). Applying the inverse operator \(L^{-1}() = \int_0^\infty \int_0^\infty \int_0^\infty dt \, dt \, dt\) to both sides of (7) and employing the boundary
condition (8) gives

\[
f = s - \eta + \frac{\eta^2}{2} + \frac{M^2 f'}{f'} + L^{-1}(f') + L^{-1}(\frac{f''}{M^2}) - mL^{-1}(ff'), 
\]
where \( x = f''(0) \) is to be determined. In ADM [17], the nonlinear terms in (11) can be decomposed as,

\[
f^2 = \sum_{k=0}^{\infty} A_k, \quad ff'' = \sum_{k=0}^{\infty} B_k.
\]

(12)

Adopting the algorithm for the Adomian polynomials proposed by Zhu et al. [32], it can be shown that

\[
A_i = \sum_{k=0}^{i} f_k f_{i-k}, \quad B_i = \sum_{k=0}^{i} f_k f_{i-k}, \quad \forall i = 0, \ldots, n.
\]

(13)

Substituting (12) into (11) yields

\[
\sum_{k=0}^{\infty} f_k = s - \eta + x \frac{\eta^2}{2} + M^2 L^{-1} f_k + L^{-1} \sum_{k=0}^{\infty} A_k - mL^{-1} \sum_{k=0}^{\infty} B_k.
\]

(14)

Hence, adopting the modified technique of Wazwaz [24], we have the simple recursive Adomian algorithm for generating the individual terms of the series solution of (7)–(9),

\[
f_0 = s - \eta,
\]

(15)

\[
f_1 = x + \frac{\eta^2}{2} + M^2 L^{-1}(f_0) + L^{-1}(A_0) - mL^{-1}(B_0),
\]

(16)

\[
f_{k+1} = M^2 L^{-1}(f_k) + L^{-1}(A_k) - mL^{-1}(B_k), \quad \forall k = 1, \ldots, n.
\]

(17)

For practical numerical computations, we shall use the finite j-term approximation of \( f(\eta) \),

\[
\phi_j(\eta) = \sum_{i=0}^{j-1} f_i.
\]

(18)

The algorithm (15)–(17) is coded in the computer algebra package Maple and we employ Maple’s built-in Padé approximants procedure. The Maple environment variable Digits controlling the number of significant digits is set to 16 in all the calculations done in this paper. To achieve reasonable accuracy we obtain the 41-term approximation of \( f(\eta) \), i.e. \( \phi_{41}(\eta) = \sum_{i=0}^{40} f_i \), where the first four terms are given as follows:

\[
f_0 = s - \eta,
\]

(19)

\[
f_1 = \frac{x}{2} \eta \eta^2 + \frac{1}{6} \left(1 - M^2\right) \eta^3,
\]

(20)

\[
f_2 = -\frac{1}{6} msx \eta^3 + \frac{1}{24} \left[ms \left(1 - M^2\right) + x \left(m - 2 + M^2\right)\right] \eta^4
\]

\[+ \frac{1}{60} \left[M^2 \left(3 - M^2\right) + m \left(1 - M^2\right) - 1\right] \eta^5,
\]

(21)

\[
f_3 = \frac{1}{24} msx^2 \eta^4 + \frac{1}{60} \left[\frac{1}{2} M^2 msx + x^2 + msx\right.
\]

\[-\frac{m}{2} \left[ms \left(1 - M^2\right) + mx - 2x + M^2 x\right] + 2msx + x^2 \left] \eta^5\right.
\]

\[+ \frac{1}{120} \left(x \left(1 - M^2\right) + \frac{3}{2} \left[ms \left(1 - M^2\right) - \frac{1}{x} \left(m + 2 - M^2\right)\right.\right.
\]

\[-m \left(\frac{3}{2} \left(M^2 - 1 + M^2 \left(1 - M^2\right) + m \left(1 - M^2\right)\right) + \frac{1}{2} \left(ms \left(1 - M^2\right)\right.\right.
\]

\[-m x + 2x - M^2 x + x \left(1 - M^2\right) \left] + \frac{1}{6} \left(1 - M^2\right) x\right.
\]

\[+ \frac{M^2}{6} \left(ms \left(1 - M^2\right) + mx - 2x + M^2 x\right) \eta^6 + \frac{1}{210} \left[\frac{1}{2} \left(1 - M^2\right)^2\right.
\]

\[\left. + \frac{M^2}{12} \left(M^2 - 1 + \frac{M^2}{2} \left(1 - M^2\right) + m \left(1 - M^2\right)\right) - \frac{1}{6} \left(M^2 + 1\right)\right.
\]

\[-\frac{M^2}{2} \left(1 - M^2\right) - m \left(1 - M^2\right)\left] - \frac{m}{3} \left(1 - M^2 - \frac{M^2}{2} \left(1 - M^2\right)\right.
\]

\[-m \left(1 - M^2\right) + \frac{1}{2} \left(1 - M^2\right) \left(1 - M^2\right)\right]\eta^7.
\]

(22)
In Table 1 we present the value of \( a = f''(0) \) at different orders of Padé approximants for \( s = 1 \), \( M = 2 \) and the cases: one-direction shrinking \((m = 1)\) and axisymmetric shrinking \((m = 2)\). Clearly the \( a \) values given in Table 1 agree very well with that of Sajid and Hayat [3] obtained by HAM. Our results for the velocity profiles presented in Figs. 1 and 2 match very well with that of Sajid and Hayat [3] depicted in their Figs. 2 and 3. In Figs. 1 and 2 we demonstrate the effects of the suction parameter \( s \) and the Hartmann number \( M \) on the velocity profiles for \( m = 1 \) and \( m = 2 \). The effect of increasing the suction parameter \( s \) and the Hartmann number \( M \) is to increase the velocity and decrease the boundary layer thickness. The above computational work may take quite a long solution time mainly due to the lengthy series solution and all roots of the Padé approximant of order \( [N/N] \) have to be searched. In addition, the computational work is also susceptible to the computer’s memory limitation and the speed of the processor used. In this work, we utilize Maple 9.5 running on a personal computer with Pentium 4 processor of 512MB RAM. Furthermore, higher order ADM series and Padé approximant are necessary for a highly nonlinear differential equation depending on the values of the parameters involved since different set of parameters values yields different degree of the strength of the nonlinearity of the resulted differential equations. Therefore, the numerical accuracy of the roots of the Padé approximants is subjected to this nonlinearity and the convergence rate will differ for every set of parameters values considered. We note that the convergence characteristics of the ADM series have been investigated by Cherruault [33] and Cherruault and Adomian [34].

4. Concluding remarks

A simple algorithm based on the ADM–Padé approach was presented for solving the MHD viscous flow due to a shrinking sheet. Our results compare very well with the results of Sajid and Hayat [3] who employed the homotopy analysis method for solving the problem.

<table>
<thead>
<tr>
<th>Padé</th>
<th>( m = 1 )</th>
<th>( m = 2 )</th>
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<tr>
<td>([5/5])</td>
<td>2.30273</td>
<td>1.33605</td>
</tr>
<tr>
<td>([10/10])</td>
<td>2.30278</td>
<td>2.89131</td>
</tr>
<tr>
<td>([15/15])</td>
<td>2.30278</td>
<td>2.89160</td>
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<td>([20/20])</td>
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</tr>
<tr>
<td>([25/25])</td>
<td>2.30278</td>
<td>2.89160</td>
</tr>
<tr>
<td>Ref. [3]</td>
<td>2.30277</td>
<td>2.89160</td>
</tr>
</tbody>
</table>

Fig. 1. The velocity profiles for \( M = 2 \) and several values of \( s \): (a) \( m = 1 \) and (b) \( m = 2 \).

Fig. 2. The velocity profiles for \( s = 1 \) and several values of \( M \): (a) \( m = 1 \) and (b) \( m = 2 \).
The ADM–Padé approach could be a promising tool for solving more complex boundary layer equations than the one studied in this paper.

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References