Variational iteration method for fractional heat- and wave-like equations

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Abstract

This paper applies the variational iteration method to obtaining analytical solutions of fractional heat- and wave-like equations with variable coefficients. Comparison with the Adomian decomposition method shows that the VIM is a powerful method for the solution of linear and nonlinear fractional differential equations.

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1. Introduction

Many phenomena in engineering physics, chemistry, other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional non-integer order [1–4]. Fractional differential equations have gained much attention recently due to exact description of nonlinear phenomena. No analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method (VIM) was first proposed to solve fractional differential equations with great success [15]. Following the above idea, Draganescu [16], Momani and Odibat [17–23] applied VIM to more complex fractional differential equations, showing the effectiveness and accuracy of the method.

In 2002 the Adomian decomposition method (ADM) was suggested to solve fractional differential equations [5]. But many researchers found it very difficult to calculate the Adomian polynomials, see [28–32]. In 2007, Momani and Odibat [33–35] applied the homotopy perturbation method (HPM) to fractional differential equations and showed that HPM is an alternative analytical method for fractional differential equations. Xu and Cang [45] solved the fractional heat- and wave-like equations with variable coefficients using homotopy analysis method (HAM).

Another powerful analytical method is called the variational iteration method (VIM) first proposed by He [8], and also see [7,9–15]. VIM has successfully been applied to many situations. For example, Soliman [24] used VIM to find a explicit solutions of KdV–Burgers’ and Lax’s seventh-order KdV equations, Batiha et al. [25] applied VIM to solve heat- and wave-like equations with singular behaviors. Furthermore, Batiha et al. [26] have expanded VIM in the form of Multistage VIM to solve a class of nonlinear system of ODEs, Wazwaz [38] applied VIM to solve linear and...
nonlinear Schrodinger equations. Shou et al. [36] solved heat-like and wave-like equations with variable coefficients by VIM, Sweilam [27] used VIM to solve multi-order FDEs, and in general. Very recently, Yu and Lib [39] solved the synchronization of fractional-order Rössler hyperchaotic systems using VIM.

In this paper, we will consider, amongst others, the three-dimensional fractional heat- and wave-like equations of the form [6]:

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}
\]

\[0 < x < a, 0 < y < b, 0 < z < c, 0 < \alpha \leq 2, t > 0,\]

subject to the boundary conditions

\[
\begin{align*}
    u(0, y, z, t) &= f_1(y, z, t), & u_x(a, y, z, t) &= f_2(y, z, t), \\
    u(x, 0, z, t) &= g_1(x, z, t), & u_y(b, y, z, t) &= g_2(x, z, t), \\
    u(x, y, 0, t) &= h_1(x, y, t), & u_z(c, y, z, t) &= h_2(x, y, t),
\end{align*}
\]

and the initial conditions

\[
u(x, y, z, 0) = \psi(x, y, z), \quad u_t(x, y, z, 0) = \eta(x, y, z),\]

where \(\alpha\) is a parameter describing the fractional derivative and \(u_t\) is the rate of change of temperature at a point over time. \(u = u(x, y, z, t)\) is temperature as a function of time and space, while \(u_{xx}, u_{yy}, \) and \(u_{zz}\) are the second spatial derivatives (thermal conductions) of temperature in \(x, y,\) and \(z\) directions, respectively. Finally, \(f(x, y, z), g(x, y, z)\) and \(h(x, y, z)\) are any functions in \(x, y\) and \(z.\)

In case \(0 < \alpha \leq 1,\) Eq. (1) reduce to the fractional heat-like equation with variable coefficients. And in case \(1 < \alpha \leq 2,\) Eq. (1) reduce to the fractional wave-like equation which models anomalous diffusive and subdiffusive systems, description of fractional random walk, unification of diffusion and wave propagation phenomena [40–43]. Recently, Eq. (1) was used to model in some fields like fluid mechanics [44]. Momani and Odibat [18] used VIM to find approximation solutions of one dimension of heat- and wave-like equations. From Momani and Odibat [6, 18] and also Shou and He [36], we study the multi-dimensional time fractional heat- and wave-like equations by using VIM.

The objective of the present paper is to extend the application of the variational iteration method (VIM) to provide approximate solution for heat- and wave-like equations of fractional order and to make comparison with that obtained by Adomian decomposition method (ADM).

2. Fractional calculus

This section is devoted to the description of the operational properties for the purpose of acquainting with sufficient fractional calculus theory, to enable us to follow the solutions of the problem given in this paper. Many definitions and studies of fractional calculus have been proposed in the last two centuries. These definitions include, Riemann–Liouville, Weyl, Reizé, Compos, Caputo, and Nashimoto fractional operators. We give some definitions and properties of the fractional calculus.

**Definition 1.** A real function \(f(x), x > 0,\) is said to be in space \(C_{\mu}, \mu \in \mathbb{R},\) if there exists a real number \(p > \mu,\) such that \(f(x) = x^p f_1(x),\) where \(f_1(x) \in C(0, \infty),\) and it is said to be in the space \(C_{\mu}^{\infty}\) if and only if \(f^{(m)} \in C_{\mu}, m \in \mathbb{N}.\)

**Definition 2.** The Riemann–Liouville fractional integral operator of order \(\alpha \geq 0\) of a functional \(f \in C_{\mu}, \mu \geq -1\) is defined as

\[
J_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, x > 0,
\]

\[
J_0^0 f(x) = f(x).
\]

Properties of the operator \(J_0^\mu\) can be found in [3], we mention the following:

For \(f \in C_{\mu}^{\infty}, \alpha, \beta > 0, \mu \geq -1\) and \(\gamma \geq -1,\)
(1) $J^a_x f(x)$ exist for almost every $x \in [a, b]$,
(2) $J^a_x J^b_x f(x) = J^{a+b}_x f(x)$,
(3) $J^a_x J^b_x f(x) = J^b_x J^a_x f(x)$,
(4) $J^a_x (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}$.

**Definition 3.** The fractional derivative of $f(x)$ in Caputo sense is defined as
\[
D^\alpha_a f(x) = J^{m-\alpha}_a D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\xi)^{m-\alpha-1} f^{(m)}(\xi) d\xi,
\]
for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $f \in C^m_m$.
(5)

Also, we need here two basic properties of the Caputo’s fractional derivative [3].

**Lemma 4.** If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C^m_m$ $\mu \geq -1$, then
\[
D^\alpha_a J^\mu_a f(x) = f(x),
\]
and
\[
J^\alpha_a D^\mu_a f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, \quad x > 0.
\]
(6)

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider multi-dimensional time fractional heat- and wave-like equations [6], where the unknown function $u = u(x, t)$ is assumed to be a casual function of time, i.e, vanishing for $t > 0$, and the fractional derivative is taken in Caputo sense to be:

**Definition 5.** For $m$ to be the smallest integer that exceeds $\alpha$, the Caputo fractional derivative of order $\alpha > 0$ is defined as
\[
D^\alpha u(x, t) = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x, \xi)}{\partial \xi^m} d\xi, & \text{for } m-1 < \alpha \leq m, \\
\frac{\partial^m u(x, t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N}.
\end{cases}
\]
(7)

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references [1–4].

3. Variational iteration method

To illustration the basic concepts of the variational iteration method (VIM) we consider the following general nonlinear system:
\[
Lu(x, y, z, t) + Nu(x, y, z, t) = g(x, y, z, t),
\]
(8)

where $L$ is the linear operator and $N$ is the nonlinear operator, and $g(x, y, z, t)$ is the inhomogeneous term. In [7–15] the variational iteration method (VIM) was proposed by He, where a correction function for Eq. (8) can be written:
\[
u_{n+1}(x, y, z, t) = \nu_n(x, y, z, t) + \int_0^t \lambda(\xi) (Lu_n(x, y, z, \xi) + Nu_n(x, y, z, \xi) - g(x, y, z, \xi)) d\xi.
\]
(9)

It is obvious that the successive approximation $u_j$, $j \geq 0$ can be established by determining $\lambda$, a general Lagrange multiplier, which can be identified optimally via the variational theory [37]. The function $\tilde{u}_n$ is a restricted variation, which means $\delta \tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier $\lambda$ that will be identified via integration by parts. The successive approximations $u_{n+1}(x, t) \geq 0$ of the solution $u(x, t)$ will be readily obtained upon using the Lagrange multiplier obtained by using any selective function $u_0(x, t)$. 

3.1. Fractional heat- and wave-like equations

First, we consider the following fractional heat-like equations of the form
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz},
\]
\[
0 < x < a, 0 < y < b, 0 < z < c, 0 < \alpha \leq 1, t > 0.
\] (10)

The variational iteration method that assumes a correction functional for Eq. (10) can be approximately expressed as follows:
\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \lambda(\xi) \left( \frac{\partial^m}{\partial \xi^m} u_n(x, y, z, \xi) - f(x, y, z)\frac{\partial^2}{\partial x^2} u_n(x, y, z, \xi) - g(x, y, z)\frac{\partial^2}{\partial y^2} u_n(x, y, z, \xi) - h(x, y, z)\frac{\partial^2}{\partial z^2} u_n(x, y, z, \xi) - \delta Q(x, y, z, \xi) \right) d\xi.
\] (11)

Making the above correction functional stationary, and noting that \(\delta u_n = 0\)
\[
\delta u_{n+1}(x, y, z, t) = \delta u_n(x, y, z, t) + \int_0^t \lambda(\xi) \left( \frac{\partial^m}{\partial \xi^m} u_n(x, y, z, \xi) - \delta Q(x, y, z, \xi) \right) d\xi.
\] (12)

For \(m = 1\) we obtain for Eq. (12) the following stationary conditions
\[
1 + \lambda(t)\xi = 0
\]
\[
\lambda'(\xi) = 0.
\] (13)

The general Lagrange multipliers, therefore, can be identified:
\[
\lambda(\xi) = -1.
\]

As a result, we obtain the following iteration formula:
\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \lambda(\xi) \left( \frac{\partial^m}{\partial \xi^m} u_n(x, y, z, \xi) - f(x, y, z)\frac{\partial^2}{\partial x^2} u_n(x, y, z, \xi) - g(x, y, z)\frac{\partial^2}{\partial y^2} u_n(x, y, z, \xi) - h(x, y, z)\frac{\partial^2}{\partial z^2} u_n(x, y, z, \xi) - \delta Q(x, y, z, \xi) \right) d\xi.
\] (14)

Second, we consider the following fractional wave-like equations of the form
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz},
\]
\[
0 < x < a, 0 < y < b, 0 < z < c, 1 < \alpha \leq 2, t > 0.
\] (15)

The variational iteration method that assumes a correction functional for Eq. (15) can be approximately expressed as follows:
\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \lambda(\xi) \left( \frac{\partial^m}{\partial \xi^m} u_n(x, y, z, \xi) - f(x, y, z)\frac{\partial^2}{\partial x^2} u_n(x, y, z, \xi) - g(x, y, z)\frac{\partial^2}{\partial y^2} u_n(x, y, z, \xi) - h(x, y, z)\frac{\partial^2}{\partial z^2} u_n(x, y, z, \xi) - \delta Q(x, y, z, \xi) \right) d\xi.
\] (16)

Making the above correction functional stationary, and noting that \(\delta u_n = 0\),
\[
\delta u_{n+1}(x, y, z, t) = \delta u_n(x, y, z, t) + \delta \int_0^t \lambda(\xi) \left( \frac{\partial^m}{\partial s^m} u_n(x, y, z, \xi) - q(x, y, z, \xi) \right) d\xi.
\] (17)

For \( m = 2 \), we obtain for Eq. (17) the following stationary conditions

\[
1 - \lambda'(t)_{|\xi=t} = 0,
\]

\[
\lambda(t)_{|\xi=t} = 0,
\]

\[
\lambda''(\xi) = 0.
\]

The general Lagrange multipliers, therefore, can be identified:

\[
\lambda(\xi) = \xi - t.
\] (18)

As a result, we obtain the following iteration formula:

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t (\xi - t) \left( \frac{\partial^m}{\partial s^m} u_n(x, y, z, \xi) - f(x, y, z) \frac{\partial^2}{\partial x^2} u_n(x, y, z, \xi) - h(x, y, z) \frac{\partial^2}{\partial z^2} u_n(x, y, z, \xi) - q(x, y, z, \xi) \right) d\xi.
\] (19)

4. Applications

In this section we shall illustrate the variational iteration method (VIM) to fractional heat-and wave-like equations.

4.1. Example 1

We consider the one-dimensional fractional heat-like equation:

\[
D_t^\alpha u = \frac{1}{2} x^2 u_{xx}, \quad 0 < x < 1, \ 0 < \alpha \leq 1, \ t > 0,
\] (20)

subject to the boundary conditions

\[
u(0, t) = 0, \quad \nu(1, t) = e^t,
\] (21)

and the initial condition

\[
u(x, 0) = x^2.
\] (22)

The exact solution (\( \alpha = 1 \)) was found to be [6]

\[
u(x, t) = x^2 e^t.
\] (23)

To solve Eq. (20) by means of VIM, we use formula of iteration (14) to find the iteration for Eq. (20) given by

\[
u_{n+1}(x, t) = \nu_n(x, t) - \int_0^t \left( \frac{\partial^\alpha}{\partial t^\alpha} \nu_n(x, \xi) - \frac{1}{2} \left[ x^2 \frac{\partial^2 \nu_n(x, \xi)}{\partial x^2} \right] \right) d\xi.
\] (24)

To get the iteration, we start with an initial approximation \( \nu_0(x, t) = \nu(x, 0) = x^2 \), that was given by Eq. (22). By using the above iteration formula (24) we can obtain the other components by using mathematical tools MAPLE package as follows:

\[
u_0(x, t) = x^2,
\] (25)

\[
u_1(x, t) = \nu_0(x, t) + x^2 t,
\] (26)

\[
u_2(x, t) = \nu_1(x, t) + \frac{t^2 x^2}{2} - \frac{x^2 t^{(2-\alpha)}}{(2-\alpha) \Gamma(2-\alpha)}.
\] (27)
\[ u_3(x,t) = u_2(x,t) + x^2 \left[ \frac{t}{6} + t^2 + \frac{t^3}{6} + \frac{t^{(3-2\alpha)}}{\Gamma(4-2\alpha)} - \frac{6t^{(2-\alpha)}}{\Gamma(2-\alpha)} + 2\alpha t^{(2-\alpha)} - \frac{2t^{(3-\alpha)}}{\Gamma(2-\alpha)} \right], \]

\[ u_4(x,t) = u_3(x,t) + x^2 \left[ \frac{t^3}{2} + \frac{t^4}{24} - t^2 + \frac{t^{(3-2\alpha)}}{\Gamma(4-2\alpha)} - \frac{6t^{(2-\alpha)}}{\Gamma(5-2\alpha)} + 12t^{(3-2\alpha)} - \frac{24t^{(3-2\alpha)}}{\Gamma(5-\alpha)} \right. \\
+ \left. \frac{3t^{(4-2\alpha)}}{\Gamma(5-\alpha)} - \frac{3t^{(2-\alpha)}}{\Gamma(5-\alpha)} + \frac{6t^{(3-\alpha)}}{\Gamma(5-\alpha)} - \frac{24t^{(3-\alpha)}}{\Gamma(5-\alpha)} + \frac{3\alpha t^{(4-\alpha)}}{\Gamma(5-\alpha)} - \frac{3\alpha t^{(2-\alpha)}}{\Gamma(5-\alpha)} - \frac{2\alpha t^{(3-\alpha)}}{\Gamma(5-\alpha)} \right], \]

and so on.

Fig. 1 shows the approximate solutions for Eq. (20) obtained for different values of \( \alpha \) using the 4-term of the variational iteration method and Fig. 2 using the 4-term of the decomposition series [6], respectively. From Table 1, we also conclude that our approximate solutions are in good agreement with the exact values. Both of VIM and ADM have highly accurate solutions, but VIM has an easier way than ADM. We can integrate the equation directly without calculating the Adomian polynomials.

Fig. 1. The surface generated from \( u_3(x,y,t) \) of variational iteration method (VIM) for one-dimensional fractional heat-like equation when \( \alpha = 1.0 \).

Table 1
Approximate solution of (20) for some values of \( \alpha \) using the 4-term VIM and ADM, respectively when \( \alpha = 0.75, 0.9, \) and \( \alpha = 1.0 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( \alpha = 0.75 )</th>
<th>( \alpha = 0.9 )</th>
<th>( \alpha = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( VIM )</td>
<td>( \phi_3 )</td>
<td>( VIM )</td>
<td>( \phi_3 )</td>
<td>( VIM )</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3</td>
<td>0.1293874976</td>
<td>0.1346451171</td>
<td>0.1210397676</td>
</tr>
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<td>0.5385804682</td>
<td>0.4841590707</td>
<td>0.4872459816</td>
</tr>
<tr>
<td>0.9</td>
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<td>1.2118060540</td>
<td>1.0893579090</td>
<td>1.0963034590</td>
</tr>
<tr>
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<td>0.3</td>
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<td>0.1798554459</td>
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</tr>
<tr>
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</tr>
<tr>
<td>0.9</td>
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</tr>
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</tr>
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</tr>
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</tr>
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</table>
4.2. Example 2

Next, we consider the two-dimensional fractional heat-like equation:

$$D_t^\alpha u = u_{xx} + u_{yy}, \quad 0 < x, y < 2\pi, \ 0 < \alpha \leq 1, \ t > 0,$$

(30)

subject to the boundary conditions

$$u(0, y, t) = 0, \quad u(2\pi, y, t) = 0$$

$$u(x, 0, t) = 0, \quad u(x, 2\pi, t) = 0$$

(31)

and the initial condition

$$u(x, y, 0) = \sin x \sin y.$$  

(32)

The exact solution ($\alpha = 1$) was found to be [6]

$$u(x, t) = e^{-2t} \sin x \sin y.$$  

(33)

To solve Eq. (30) by means of VIM, we use formula of iteration (14) to find the iteration for Eq. (30) given by

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left( \frac{\partial^\alpha}{\partial t^\alpha} u_n(x, y, \xi) - \frac{1}{2} \left[ y^2 \frac{\partial^2 u_n(x, y, \xi)}{\partial x^2} + x^2 \frac{\partial^2 u_n(x, y, \xi)}{\partial y^2} \right] \right) \, d\xi.$$  

(34)

To get the iteration, we start with an initial approximation $u_0(x, y, t) = u(x, y, 0) = \sin x \sin y$, that was given by Eq. (32), we obtain the following successive approximations as follows:

$$u_0(x, y, t) = \sin x \sin y,$$

(35)

$$u_1(x, y, t) = (1 - 2t) \sin x \sin y,$$

(36)

$$u_2(x, y, t) = \left( 1 - 4t + 2t^2 + \frac{2t(\alpha - 2)}{\Gamma(3 - \alpha)} \right) \sin x \sin y,$$

(37)
\[ u_3(x, y, t) = \left( 1 - 6t + 6t^2 - \frac{4}{3} t^3 - \frac{1}{\Gamma(4-\alpha)} \left[ 6\alpha t^{(4-\alpha)} - 8t^{(3-\alpha)} + 18t^{(2-\alpha)} \right] - \frac{2t^{(3-5\alpha)}}{\Gamma(4-2\alpha)} \right) \sin x \sin y, \quad (38) \]

and so on.

The result can be seen in Figs. 3 and 4 and Table 2. Fig. 3 shows the approximate solutions of Eq. (30) obtained for different values of \( \alpha \) using the 4-term of the variational iteration method and Fig. 4 using the 4-term of the decomposition series [6], respectively. From Table 2, we also conclude that our approximate solutions are in good agreement with the exact values. Both of VIM and ADM have highly accurate solutions, but VIM has an easier way than ADM. We can integrate the equation directly without calculating the Adomian polynomials.

![Fig. 3](image1.png)

**Fig. 3.** The surface generated from \( u_3(x, y, t) \) of variational iteration method (VIM) for two-dimensional fractional heat-like equation with \( \alpha = 1.0 \).

![Fig. 4](image2.png)

**Fig. 4.** The surface generated from 4-term of Adomian decomposition method (VIM) for two-dimensional fractional heat-like equation with \( \alpha = 1.0 \).
Consider the following three-dimensional fractional heat-like equation:

\[
D_t^\alpha u = x^4 y^4 z^4 + \frac{1}{36} \left[ x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} \right],
\]

subject to the boundary conditions

\[
\begin{align*}
&u(0, y, z, t) = 0, \quad u(1, y, z, t) = y^4 z^4 (e^t - 1), \\
&u(x, 0, z, t) = 0, \quad u(x, 1, z, t) = z^4 (e^t - 1), \\
&u(x, y, 0, t) = 0, \quad u(x, y, 1, t) = x^4 y^4 (e^t - 1),
\end{align*}
\]

and the initial condition

\[
u(x, y, z, 0) = 0.
\]

The exact solution \((\alpha = 1)\) was found to be [6]

\[
u(x, y, z, t) = x^4 y^4 z^4 (e^t - 1).
\]

To solve Eq. (39) by means of VIM, we use formula of iteration (14) to find the iteration for Eq. (39) given by:

\[
u_{n+1}(x, y, t) = \nu_n(x, y, t) - \int_0^t \left( \frac{\partial^{\alpha}}{\partial t^{\alpha}} \nu(x, y, \xi) - x^4 y^4 z^4 \right) \left[ x^2 \frac{\partial^2 \nu(x, y, \xi)}{\partial x^2} + y^2 \frac{\partial^2 \nu(x, y, \xi)}{\partial y^2} + z^2 \frac{\partial^2 \nu(x, y, \xi)}{\partial z^2} \right] d\xi.
\]

To get the iteration, we start with an initial approximation \(\nu_0(x, y, t) = x^4 y^4 z^4 \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha\), that was given by [16], we obtain the following successive approximations as follows:

\[
\begin{align*}
u_0(x, y, z, t) &= x^4 y^4 z^4 \frac{\Gamma(1)}{\Gamma(\alpha+1)} t^\alpha, \\
u_1(x, y, z, t) &= \nu_0(x, y, z, t) + \frac{x^4 y^4 z^4 t^{\alpha+1}}{\Gamma(\alpha+2)}, \\
u_2(x, y, z, t) &= \nu_1(x, y, z, t) + x^4 y^4 z^4 t,
\end{align*}
\]
Fig. 5 shows the approximate solutions for Eq. (39) obtained for differential value of $\alpha$ using 2-term of VIM and Fig. 6 using 2-term of ADM [6], respectively. From Table 3, we conclude that VIM is more accurate than ADM [6]. VIM value approaches the exact solution value, when $\alpha = 2.0$.

\begin{align}
\sum_{n=0}^{\infty} M_{n+1} & = \sum_{n=0}^{\infty} \frac{1}{\alpha^n} \\
& = \frac{1}{1 - \frac{1}{\alpha}} \\
& = \frac{\alpha}{\alpha - 1}
\end{align}

and so on.

\textbf{Table 3}

Approximate solution of (39) for some values of $\alpha$, that is $\alpha = 0.5, 0.75, \text{ and } \alpha = 1$ by using the 2-term VIM and first-order term of Adomian decomposition method [16]

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.75$</th>
<th>$\alpha = 1.0$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>VIM</td>
<td></td>
<td></td>
<td>ADM $\phi_1$</td>
<td>VIM $\phi_1$</td>
<td>ADM $\phi_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(47)</td>
<td>(47)</td>
<td>(47)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.25</td>
<td>0.1</td>
<td>0.3</td>
<td>0.00000000020</td>
<td>0.00000146276</td>
<td>0.00000000012</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td></td>
<td>0.3</td>
<td>0.0000000316</td>
<td>0.00002340421</td>
<td>0.0000000194</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td></td>
<td>0.3</td>
<td>0.0000012351</td>
<td>0.00000146276</td>
<td>0.00000007564</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td></td>
<td>0.3</td>
<td>0.0000197620</td>
<td>0.00002340421</td>
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<td>0.5</td>
<td>0.00000146276</td>
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<td>0.00000000012</td>
</tr>
</tbody>
</table>

(continued on next page)
4.4. Example 4

Next, we consider the two-dimensional fractional wave-like equation:

\[ D_\gamma^{\alpha}u = \frac{1}{12}(x^2u_{xx} + y^2u_{yy}), \quad 0 < x, y < 1, \; 1 < \alpha \leq 2, \; t > 0, \]

subject to the boundary conditions

\[ u(0, y, t) = 0 \quad u(1, y, t) = 4 \cosh t, \]

\[ u(x, 0, t) = 0 \quad u(x, 1, t) = 4 \sinh t, \]

and the initial condition

\[ u(x, y, 0) = x^4, \quad u_t(x, y, 0) = y^4. \]

The exact solution (\( \alpha = 2 \)) was found to be [36]:

\[ u(x, y, t) = x^4 \cosh t + y^4 \sinh t. \]

To solve Eq. (48) by means of VIM, we use formula of iteration (19) to find the iteration for Eq. (48) given by:

\[
\begin{align*}
  u_{n+1}(x, y, t) &= u_n(x, y, t) + \int_0^t (\xi - t) \left( \sum_{k=0}^{n-1} \left[ y^2 \frac{\partial^2 u_n(x, y, \xi)}{\partial x^2} + x^2 \frac{\partial^2 u_n(x, y, \xi)}{\partial y^2} \right] \right) d\xi.
\end{align*}
\]
To get the iteration, we start with an initial approximation $u_0(x, y, t) = u(x, y, 0) = x^4 + y^4 t$, that was given by [36]. We obtain the following successive approximations as follows:

$$u_0(x, y, t) = x^4 + y^4 t,$$

$$u_1(x, y, t) = u_0(x, y, t) + \frac{1}{6} y^4 t^3 + \frac{1}{2} x^4 t^2,$$

$$u_2(x, y, t) = x^4 + y^4 t + \frac{1}{3} y^4 t^3 + \frac{1}{2} t^2 x^4 + \frac{t^4 y^4}{24} + \frac{t^5 y^4}{120} + \frac{x^4 t^{(5-\alpha)}}{\Gamma(5-\alpha)(5-\alpha)} - \frac{y^4 t^{(4-\alpha)}}{\Gamma(4-\alpha)(4-\alpha)},$$

and so on.

Figs. 7 and 8 show the comparison between 3-term the variational iteration method (VIM) and the Adomian decomposition method (ADM) for different values of $\alpha$, respectively. From Table 4, the approximation solution of Eq. (48) shows that variational iteration method (VIM) is more accurate than the Adomian decomposition method (ADM) when $\alpha = 2.0$.

---

Table 4

Approximate solution of (48) for some values of $\alpha$, that is $\alpha = 1.5$, 1.75, and $\alpha = 2.0$ using the 3-term VIM and ADM, respectively

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
<th>$\alpha = 1.5$</th>
<th>$\alpha = 1.75$</th>
<th>$\alpha = 2.0$</th>
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<tr>
<td></td>
<td></td>
<td></td>
<td>VIM $u_2$</td>
<td>ADM $\phi_3$</td>
<td>VIM $u_2$</td>
</tr>
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<tr>
<td></td>
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</tr>
<tr>
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<td>0.05</td>
<td>0.000796499</td>
<td>0.00986331</td>
<td>0.006392797</td>
</tr>
<tr>
<td>0.5</td>
<td>0.01</td>
<td>0.25</td>
<td>0.00208723</td>
<td>0.16268483</td>
<td>0.06532664</td>
</tr>
<tr>
<td></td>
<td>0.05</td>
<td>0.25</td>
<td>0.00208723</td>
<td>0.16268483</td>
<td>0.06532664</td>
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<tr>
<td></td>
<td>0.25</td>
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<td>0.000365782</td>
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</tbody>
</table>

---

Fig. 7. The surface generated from $u_2(x, y, t)$ of variational iteration method (VIM) for two-dimensional fractional wave-like equation with $\alpha = 2.0$. 
Fig. 8. The surface generated from 3-term Adomian decomposition method (ADM) for two-dimensional fractional wave-like equation with \( \alpha = 2.0 \).

4.5. Example 5

Finally, we consider the three-dimensional fractional wave-like equation:

\[
D^\alpha_{tt} u = x^2 y^2 z^2 + \frac{1}{2} \left[ x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} \right],
\]

\( 0 < x, y, z < 1, \ t > 0, \ 1 < \alpha \leq 2 \),

subject to the boundary conditions

\[
\begin{align*}
\text{at } t = 0: & \quad u(0, y, z, t) = y^2 (e^t - 1) + z^2 (e^{-t} - 1), \\
\text{at } t = 1: & \quad u(1, y, z, t) = (1 + y^2)(e^t - 1) + z^2 (e^{-t} - 1), \\
\text{at } x = 0: & \quad u(x, 0, z, t) = x^2 (e^t - 1) + z^2 (e^{-t} - 1), \\
\text{at } x = 1: & \quad u(x, 1, z, t) = (1 + x^2)(e^t - 1) + y^2 (e^{-t} - 1), \\
\text{at } y = 0: & \quad u(x, y, 0, t) = x^2 (e^t - 1) + y^2 (e^{-t} - 1), \\
\text{at } y = 1: & \quad u(x, y, 1, t) = (1 + x^2)(e^t - 1) + y^2 (e^{-t} - 1), \\
\text{and initial condition: } & \quad u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = x^2 + y^2 - z^2.
\end{align*}
\]

The exact solution \( (\alpha = 2) \) was found to be [6]

\[
u(x, y, z) = -(x^2 + y^2 + z^2) + (x^2 + y^2)e^{-t} + z^2 e^{-t}.
\]

To solve Eq. (56) by means of VIM, we use formula of iteration (19) to find the iteration for Eq. (56) given by:

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t (\xi - t) \left( \frac{\partial^{\alpha}}{\partial t^{\alpha}} u_n(x, y, z, \xi) - x^2 - y^2 - z^2 \right.
\]

\[- \frac{1}{2} \left[ x^2 \frac{\partial^2 u_n(x, y, z, \xi)}{\partial x^2} + y^2 \frac{\partial^2 u_n(x, y, z, \xi)}{\partial y^2} + z^2 \frac{\partial^2 u_n(x, y, z, \xi)}{\partial z^2} \right] \right) d\xi.
\]

To get the iteration, we start with an initial approximation

\[
u_0(x, y, t) = (x^2 + y^2) \left( t + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) + z^2 \left( -t + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right).
\]
that was given by [16]. We obtain the following successive approximations as follows:

\[ u_0(x, y, z, t) = (x^2 + y^2) \left( t + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right) + z^2 \left( -t + \frac{t^\alpha}{\Gamma(\alpha + 1)} \right), \tag{61} \]

\[ u_1(x, y, z, t) = u_0(x, y, z, t) + (x^2 + y^2 - z^2) \frac{t^3}{6} + (x^2 + y^2 + z^2) \frac{t^{(2+\alpha)}}{\Gamma(3 + \alpha)}, \tag{62} \]

and so on.

Fig. 9 shows that the approximate solutions for Eq. (56) obtained for different values of \( \alpha \) by the variational iteration method (VIM) and Fig. 10 using 2-term of the Adomian decomposition method [6], respectively. From Table 5 we conclude that variational iteration method (VIM) is more accurate than Adomian decomposition method.

Fig. 9. The surface generated from \( u_1(x, y, t) \) of variational iteration method (VIM) for three-dimensional fractional wave-like equation with \( \alpha = 2.0 \).

Fig. 10. The surface generated from 2-term Adomian decomposition method for three-dimensional fractional wave-like equation with \( \alpha = 2.0 \).
5. Conclusions

In this paper, variational iteration method (VIM) has been successfully employed to obtain the approximate solution of the fractional heat- and wave-like equations with variable coefficients. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. Finally, the recent appearance of fractional differential equations as models in some fields of applied mathematics make it necessary to investigate methods of solution for such equations (analytical and numerical) and we hope that this work is a step in this direction.

References


